16

MODAL LOGIC IN MATHEMATICS

Sergei Artemov

1 INTRODUCTION

Formal modal logic is mostly mathematical in its methods, regardless of area of application. This Handbook presents a wide variety of mathematical techniques developed over decades of studying the intricate details of modal logic. Also included among relatively recent general purpose sources on the mathematics of modal logic are monographs [57, 75, 99, 114, 153] and a survey paper [115]. For that matter, the applications of mathematics in modal logic are overwhelming, while those in the dual category, the uses of modal logic in mathematics, are less numerous.

Mathematics normally finds a proper language and level of abstraction for the study of its objects. Propositional modal logic offers a new paradigm of applying logical methods: instead of using the traditional languages with quantification (first-order or higher-order) to describe a structure, look for an appropriate quantifier-free language with additional logic operators (modalities) that represent the phenomenon at hand. In a number of prominent cases, we end up with a logic-based language which is much richer than Boolean logic, and yet, unlike universal languages with quantification, does not fall under the scope of classical undecidability limitations. Modal logic often offers better decidability and complexity results than the rival first-order logic.

We adopt a strict approach as to what constitutes an application of modal logic in mathematics, i.e., we limit our attention to mathematical objects which existed independently of modal logic, rather than those developed for the needs of modal logic itself. This requirement is not by any means sufficient; after all, any class of binary relations in mathematics specifies some propositional modal logic which, however, does not automatically make these connections worthy of study. We consider only the cases in which a mathematical modality-like notion was developed and studied by mathematicians to the extent that the modal logical language and methods became pertinent. Neither is this requirement necessary; for example, elaborate algebraic models originally developed for the needs of logic (e.g., modal logic) are now deeply embedded into the corresponding field of mathematics and may well be regarded as a contribution of modal logic to mathematics. Fortunately, algebraic models for modal logic have been covered in Chapter 6 of this Handbook. Moreover, the present author has not been quite pedantic in carrying out even this imperfect approach; such important issues as topos models and the connection between modal logic and Grothendieck topology on categories were barely mentioned in this survey. Some of these topics were considered in Chapter 9 of this Handbook.

There are two major ideas that dominate the landscape of modal logic application in mathematics: Gödel's provability semantics and Tarski's topological semantics.

Gödel's use of modal logic to describe provability in the 1930s gave the first exact semantics of modality. This approach led to a comprehensive provability semantics for a broad class of modal logics, including the major ones: K, T, $\mathsf{K4}$, $\mathsf{S4}$, $\mathsf{S5}$, GL, Grz, and others. It also proved vital for such applications as the Brouwer-Heyting-Kolmogorov (intended) provability semantics for intuitionistic logic, for introducing justification into formal epistemology and tackling its logical omniscience problem, for introducing self-reference into combinatory logic and lambda-calculi, etc.

Another major use of modal logic in mathematics is the topological semantics suggested by Tarski and developed by Tarski and McKinsey in the 1940s. Here modal logic provides a natural high-level language for describing topology in a point-free manner. In addition to its natural mathematical appeal, this approach has evolved into an active research area with applications in dynamic systems, control systems, spatio-temporal reasoning, etc.

There has also been significant research activity in applying modal logic to set theory, which can be traced back to Solovay's work of the 1970s. We devote Section 7 to this issue.

The reader might perceive a certain bias towards provability logic in this survey. A possible explanation is that Gödel's provability semantics of modal logic is the oldest and arguably the most well-established tradition of applying modal logic to mathematics. It is perhaps more essential for proof theory and foundations than other applications of modal logic for the corresponding object areas of mathematics. This observation is not intended to discount other interpretations of modal logic considered here; we hope that this survey gives a fair assessment of their beauty and vast potential.

Among other recent surveys in this area, we recommend the article 'Provability logic' by Verbrugge in the Stanford Encyclopedia of Philosophy

`http://plato.stanford.edu/entries/logic-provability/`,

the handbook chapter 'Provability Logic' [25], and the forthcoming collection 'The Logic of Space' edited by Aiello, van Benthem, and Pratt-Hartmann.

2 SOME HISTORY

In his 1933 paper [109], Gödel chose the language of propositional modal logic to describe the basic logical laws of provability. According to his approach, $\Box F$ should be interpreted informally as

F is provable,

and the classical modal logic S4 provides a system of plausible postulates for provability. Gödel's goal was to provide an exact interpretation of intuitionistic propositional logic within a classical modal logic of provability, hence giving classical meaning to the basic intuitionistic logical system.

This line of research attracted a great deal of attention in mathematics and eventually led to two distinct models of provability based on modal logics:

1. the Provability Logic GL, which was shown by Solovay to be the logic of Gödel's formal provability predicate;

2. Gödel's original logic S4, which was shown by Artemov to be a forgetful projection of the Logic of Proofs LP.

These two models complement each other and cover a wide range of applications, from traditional proof theory to formal verification and epistemology.

The use of modal logic in topology was initially motivated by Kuratowski's axioms for topological spaces, which were recast in the manner of modal logic by Tarski in the late 1930s. Under this interpretation, the Boolean components were treated in the usual set theoretical way as subsets of a given topological set, whereas $\Box$ was interpreted as

the interior operator.